

A perturbation and generic smoothness of the Vafa–Witten moduli spaces on closed symplectic four-manifolds

Yuuji Tanaka

Abstract

We formulate a Freed–Uhlenbeck style generic smoothness theorem for the moduli space of solutions to the Vafa–Witten equations on a closed symplectic four-manifold by using a method developed by Feehan for the study of the $PU(2)$ -monopole equation on smooth closed four-manifolds. We introduce a set of perturbation terms to the Vafa–Witten equations, but not depending on connections, and prove that the moduli space of solutions to the perturbed Vafa–Witten equations for the structure group being $SU(2)$ or $SO(3)$ on a closed symplectic four-manifold is a smooth manifold of dimension zero for a generic choice of the perturbation parameters.

1 Introduction

In this article, we consider the Vafa–Witten equations ([VW], [Ma], [Ha], [W]) on a compact symplectic four-manifold, and prove a Freed–Uhlenbeck type transversality theorem for the moduli space of solutions to the Vafa–Witten equations by using a method developed by Feehan [F] for the study of the $PU(2)$ -monopole equation on closed four-manifolds. We introduce a set of perturbation terms, which does not depend upon connections, and prove that the moduli space of solutions to the perturbed equations is a smooth manifold of dimension zero for a generic choice of perturbation parameters.

The Vafa–Witten equations. Let us introduce first the equations in the original form. Let X be a closed, oriented, smooth Riemannian four-manifold with Riemannian metric g , and let $P \rightarrow X$ be a principal G -bundle over X with G being a compact Lie group. We denote by \mathcal{A}_P the set of all connections of P , and by $\Omega^+(X, \mathfrak{g}_P)$ the set of self-dual two-forms valued in

the adjoint bundle \mathfrak{g}_P of P . We consider the following equations for a triple $(A, B, \Gamma) \in \mathcal{A}_P \times \Omega^+(X, \mathfrak{g}_P) \times \Omega^0(X, \mathfrak{g}_P)$.

$$d_A \Gamma + d_A^* B = 0, \quad (1.1)$$

$$F_A^+ + \frac{1}{8}[B, B] + \frac{1}{2}[B, \Gamma] = 0, \quad (1.2)$$

where $[B, B] \in \Omega^+(X, \mathfrak{g}_P)$ is defined by a point-wise Lie-algebraic structure on Λ^+ together with the bracket of \mathfrak{g}_P (see [Ma, §A.1] or [T1, §2] for more detail). We call these equations the *Vafa–Witten equations*. The equations (1.1) and (1.2) with a gauge fixing equation form an elliptic system with the index always being zero.

The equations on compact symplectic four-manifolds and a perturbation. We rewrite the equations (1.1) and (1.2) when the underlying manifold X is a compact symplectic four-manifold.

Let X be a compact symplectic four-manifold with symplectic form ω . We take an almost complex structure J compatible with the symplectic form ω . In this setting, the equations (1.1) and (1.2) can be written as follows (see Section 2).

$$\begin{aligned} \bar{\partial}_A \alpha + \bar{\partial}_A^* \beta &= 0, \\ F_A^{0,2} + \frac{1}{2}[\alpha, \beta] &= 0, \quad \omega^2 \wedge \left(i\Lambda F_A^{1,1} + \frac{1}{2}[\alpha, \alpha^*] \right) + [\beta, \beta^*] = 0, \end{aligned}$$

where $\Lambda := (\wedge \omega)^*$, and $\alpha \in \Omega^{0,0}(X, \mathfrak{g}_P)$, $\beta \in \Omega^{0,2}(X, \mathfrak{g}_P)$.

We then introduce the following perturbation for the Vafa–Witten equations on a compact symplectic four-manifold above.

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta + \rho(\theta)(\alpha + \beta) = 0, \quad (1.3)$$

$$F_A^{0,2} + \frac{1}{2}\tau_1[\alpha, \beta] = 0, \quad \omega^2 \wedge \left(i\Lambda F_A^{1,1} + \frac{1}{2}\tau_2[\alpha, \alpha^*] \right) + \tau_3[\beta, \beta^*] = 0, \quad (1.4)$$

where $\rho : T^*X \otimes \mathbb{C} \rightarrow \text{Hom}_{\mathbb{C}}(\Lambda^{0,0} \oplus \Lambda^{0,2}, \Lambda^{0,1})$ is the Clifford multiplication, and $\tau_1 \in C^r(GL(\Lambda^{0,2}))$, $\tau_2 \in C^r(GL(\Lambda^{0,0}))$, $\tau_3 \in C^r(GL(\Lambda^{2,2}))$ and $\theta \in T^*X \otimes \mathbb{C}$ are perturbation parameters. We put $\tau := (\tau_1, \tau_2, \tau_3)$. We denote by \mathcal{P}_1 the Banach space of the perturbation parameters (τ, θ) , namely, $\mathcal{P}_1 := C^r(GL(\Lambda^{0,2})) \times C^r(GL(\Lambda^{0,0})) \times C^r(GL(\Lambda^{2,2})) \times C^r(\Lambda^1 \otimes \mathbb{C})$.

This perturbation does not depend upon connections. Hence, one does not need to be careful about the compatibility with the bubbling out of connections. We prove a Freed–Uhlenbeck style transversality theorem for the moduli space of solutions to the equations (1.3) and (1.4).

Generic smoothness of the moduli spaces. Before stating results in this article, let us introduce some terminology here first.

Definition 1.1. A connection A of a principal G -bundle over X is said to be *irreducible* if the stabilizer Z_A in \mathcal{G}_P coincides with the center of the group G , and *reducible* otherwise.

We also introduce the following notion of *rank* for sections.

Definition 1.2. We say a \mathfrak{g}_P -valued form $\alpha + \beta \in \Gamma(\mathfrak{g}_P \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2}))$ is of *rank* r if, when considered as a section of $\text{Hom}((\Lambda^{0,0} \oplus \Lambda^{0,2})^*, \mathfrak{g}_P)$, the section $(\alpha + \beta)(x)$ has rank less than or equal to r at every point $x \in X$ with equality at some point.

We then denote by $\mathcal{M}_\diamond^*(\tau, \theta)$ the moduli space of solutions $(A, (\alpha, \beta))$ to the perturbed Vafa–Witten equations (1.3) and (1.4) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $\alpha + \beta$ being of rank three. Firstly, we prove the following in Section 3.

Proposition 1.3. *Let X be a closed symplectic four-manifold, and let $P \rightarrow X$ be a principal G -bundle over X with G being either $SU(2)$ or $SO(3)$. Then there is a first category subset $\mathcal{P}'_1 \subset \mathcal{P}_1$ such that for all $(\tau, \theta) \in \mathcal{P}_1 \setminus \mathcal{P}'_1$, the moduli space $\mathcal{M}_\diamond^*(\tau, \theta)$ is a smooth manifold of dimension zero.*

Here, a subset of S' of a topological space S is said to be a *first category subset* if S' is a countable union of closed subsets of S with empty interior. We mean a *generic choice* of elements in S by taking an element from such $S \setminus S'$.

We next consider rank ≤ 2 irreducible solutions to the equations. We formulate that there are no such solutions for a generic choice of perturbation parameters. In order to do this, we further perturb the equations, which corresponds to moving metrics or almost complex structures of the underlying manifold. More precisely, we introduce an extra perturbation parameter $f \in C^r(GL(T^*X))$, and consider the following equations.

$$\bar{\partial}_{A,f}\alpha + \bar{\partial}_{A,f}^*\beta + \rho(f(\theta))(\alpha + \beta) = 0, \quad (1.5)$$

$$P_f^{0,2}(F_A) + \frac{1}{2}\tau_1[\alpha, \beta] = 0, \quad \omega^2 \wedge \left(i\Lambda P_f^{1,1}(F_A) + \frac{1}{2}\tau_2[\alpha, \alpha^*] \right) + \tau_3[\beta, \beta^*] = 0, \quad (1.6)$$

where $P_f^{0,2}$ and $P_f^{1,1}$ are the projections to $(0, 2)$ and $(1, 1)$ -parts with respect to the almost complex structure f^*J , and

$$\bar{\partial}_{A,f} = \sum f(v^i) \wedge \nabla_{A,v_i}, \quad \bar{\partial}_{A,f}^* = - \sum \iota(f(v^i)) \nabla_{A,v_i},$$

where $\{v^i\}$ is an orthonormal frame of $\Lambda^{0,1}$, and $\{v_i\}$ is its dual. These $\bar{\partial}_{A,f}$ and $\bar{\partial}_{A,f}^*$ can be seen as a variation of the Dirac operator under the moving of metrics or almost complex structures of the underlying manifold.

We then put $\mathcal{P}_2 := C^r(GL(T^*X)) \times C^r(\Lambda^1 \otimes \mathbb{C})$ as the perturbation parameter space. We denote by $\mathcal{M}^{*,0}(f, \theta)$ the moduli space of solutions $(A, (\alpha, \beta))$ to the equations (1.5) and (1.6) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $(\alpha, \beta) \neq 0$. We then prove the following in Section 4.

Proposition 1.4. *Let X be a closed symplectic four-manifold, and let $P \rightarrow X$ be a principal G -bundle over X with G being either $SU(2)$ or $SO(3)$. Then there is a first category subset $\mathcal{P}'_2 \subset \mathcal{P}_2$ such that for all $(f, \theta) \in \mathcal{P}_2 \setminus \mathcal{P}'_2$, the moduli space $\mathcal{M}^{*,0}(f, \theta)$ contains no solutions $(A, (\alpha, \beta))$ to the perturbed Vafa–Witten equations (1.5) and (1.6) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $\alpha + \beta$ being of rank one or two.*

A proof here for Proposition 1.4 invokes a series of ideas by Feehan [F] in the study of the $PU(2)$ -monopole equation, which in particular uses a version of the Sard–Smale theorem (see Section 4.1).

We now take $\mathcal{P} = C^r(GL(T^*X)) \times C^r(GL(\Lambda^{0,2})) \times C^r(GL(\Lambda^{0,0})) \times C^r(GL(\Lambda^{2,2})) \times C^r(\Lambda^1 \otimes \mathbb{C})$ as the perturbation parameter space. Combining Propositions 1.3 and 1.4 above, we obtain the following.

Theorem 1.5. *Let X be a closed symplectic four-manifold, and let $P \rightarrow X$ be a principal G -bundle over X with G being either $SU(2)$ or $SO(3)$. We denote by $\mathcal{M}^{*,0}(f, \tau, \theta)$ the moduli space of solutions $(A, (\alpha, \beta))$ to the perturbed Vafa–Witten equations (1.5) and (1.6) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $(\alpha, \beta) \neq 0$. Then there is a first category subset $\mathcal{P}' \subset \mathcal{P}$ such that for all $(f, \tau, \theta) \in \mathcal{P} \setminus \mathcal{P}'$, the moduli space $\mathcal{M}^{*,0}(f, \tau, \theta)$ is a smooth manifold of dimension zero.*

In the above theorem, note that the C^r -perturbation parameter space \mathcal{P} and its first category subset can be replaced by C^∞ -perturbation parameter space and its first category subset by using an argument by Feehan–Leness [FL, §5.1.2].

2 Perturbations

In Section 2.1, we recall the description of $Spin^c$ -structures and the Dirac operators on compact symplectic manifolds. We then describe the perturbations to the equations on compact symplectic four-manifolds in Sections 2.2 and 2.3.

2.1 Spinor bundles and the Dirac operator on symplectic manifolds

A general reference for $Spin^c$ -structures and the Dirac operators is [LM].

Spinor bundles. Any almost complex manifold X has the canonical $Spin^c$ -structure associated with the almost complex structure, whose spinor bundle is $\oplus_i \Omega^{0,i}(X)$.

Any spinor bundle S splits as into the direct sum of vector bundles S^+ and S^- , where S^+, S^- are the eigenspaces of the Clifford element of ± 1 eigenvalues respectively. For X being an oriented smooth four-manifold with $Spin^c$ -structure, we have the following isomorphism induced from the Clifford multiplication.

$$T^*X \otimes \mathbb{C} \cong \text{Hom}_{\mathbb{C}}(S^+, S^-).$$

See [Mo] (or [F, A.3]) for a proof. In our setting, the almost complex case, this isomorphism can be stated as

$$T^*X \otimes \mathbb{C} \cong \text{Hom}_{\mathbb{C}}(\Lambda^{0,0} \oplus \Lambda^{0,2}, \Lambda^{0,1}).$$

The Dirac operator on symplectic manifolds. Let E be a vector bundle on X . The Dirac operator D_A associated to a connection A on E is given by the composition:

$$\Gamma(S) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes (S \otimes E)) \xrightarrow{\text{metric}} \Gamma(TX \otimes (S \otimes E)) \xrightarrow{\rho} \Gamma(S \otimes E),$$

where ρ is the Clifford multiplication.

In the almost complex case, the Dirac operator is written as

$$D_A = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*),$$

where A is a connection on E . Thus, if the underlying manifold X is a symplectic four-manifold, the Dirac equations become $\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0$, where $\alpha \in \Omega^{0,0}(E)$, $\beta \in \Omega^{0,2}(E)$.

2.2 The equations on symplectic four-manifolds and a perturbation

Let X be a compact symplectic four-manifold with symplectic form ω , and let P be a principal G -bundle over X , where G is a compact Lie group. We take an almost complex structure J compatible with the symplectic form ω .

Let us rewrite the equations (1.1) and (1.2), when the underlying manifold is a compact symplectic four-manifold. This was substantially described by Mares [Ma, §7]. We follow his notations. Firstly, we denote an orthonormal frame of Λ^1 by $\{e^0, e^1, e^2, e^3\}$. We define $dz^1 = e^0 + ie^1$, $dz^2 = e^2 + ie^3$. Here we have $\omega = e^0 \wedge e^1 + e^2 \wedge e^3$. We write $B \in \Omega^+(\mathfrak{g}_P)$ as $B = B_1(e^0 \wedge e^1 + e^2 \wedge e^3) + B_2(e^0 \wedge e^3 + e^3 \wedge e^1) + B_3(e^0 \wedge e^3 + e^1 \wedge e^2)$. We then define $\alpha \in \Omega^{0,0}(X, \mathfrak{g}_P)$ and $\beta \in \Omega^{0,2}(X, \mathfrak{g}_P)$ by

$$\alpha = \Gamma + iB_1, \quad \beta = -\frac{1}{2}(B_2 + iB_3)d\bar{z}^1 \wedge d\bar{z}^2.$$

Note that B can be written as $B = B_1\omega + \beta + \beta^*$. Note also that $\alpha - \bar{\alpha} = 0$ if A is irreducible as $\Gamma = 0$ in this case.

With these notations, the equations (1.1) and (1.2) are rewritten as follows.

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0, \quad (2.1)$$

$$F_A^{0,2} + \frac{1}{2}[\alpha, \beta] = 0, \quad \omega^2 \wedge \left(i\Lambda F_A^{1,1} + \frac{1}{2}[\alpha, \alpha^*] \right) + [\beta, \beta^*] = 0, \quad (2.2)$$

where $\Lambda := (\wedge\omega)^*$.

Perturbation. We consider the following perturbed Vafa–Witten equations.

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta + \rho(\theta)(\alpha + \beta) = 0, \quad (2.3)$$

$$F_A^{0,2} + \frac{1}{2}\tau_1[\alpha, \beta] = 0, \quad \omega^2 \wedge \left(i\Lambda F_A^{1,1} + \frac{1}{2}\tau_2[\alpha, \alpha^*] \right) + \tau_3[\beta, \beta^*] = 0, \quad (2.4)$$

where $\rho : T^*X \otimes \mathbb{C} \rightarrow \text{Hom}_{\mathbb{C}}(\Lambda^{0,0} \oplus \Lambda^{0,2}, \Lambda^{0,1})$ is the Clifford multiplication, and $\tau_1 \in C^r(GL(\Lambda^{0,2}))$, $\tau_2 \in C^r(GL(\Lambda^{0,0}))$, $\tau_3 \in C^r(GL(\Lambda^{2,2}))$ and $\theta \in T^*X \otimes \mathbb{C}$ are perturbation parameters.

As mentioned in Introduction, this perturbation does not involve connections. In Section 3, we prove that the moduli space of solutions to the equations (2.3) and (2.4) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $\alpha + \beta$ being of rank three is a smooth manifold of dimension zero for a generic choice of the perturbation parameters.

2.3 A further perturbation

Following Feehan [F, §3], we consider a perturbation of the Dirac operator. We take $f \in C^r(GL(T^*X))$, and consider the following.

$$\bar{\partial}_{A,f} = \sum f(v^i) \wedge \nabla_{A,v_i}, \quad \bar{\partial}_{A,f}^* = -\sum \iota(f(v^i)) \nabla_{A,v_i},$$

where $\{v^i\}$ is an orthonormal frame of $\Lambda^{0,1}$ and $\{v_i\}$ is its dual. These $\bar{\partial}_{A,f}$ and $\bar{\partial}_{A,f}^*$ can be seen as a variation of the Dirac operator under the moving of metrics or almost complex structures of the underlying manifold.

We then consider the following equations.

$$\bar{\partial}_{A,f}\alpha + \bar{\partial}_{A,f}^*\beta + \rho(f(\theta))(\alpha + \beta) = 0, \quad (2.5)$$

$$P_f^{0,2}(F_A) + \frac{1}{2}\tau_1[\alpha, \beta] = 0, \quad \omega^2 \wedge \left(i\Lambda P_f^{1,1}(F_A) + \frac{1}{2}\tau_2[\alpha, \alpha^*] \right) + \tau_3[\beta, \beta^*] = 0, \quad (2.6)$$

where $\theta \in T^*X \otimes \mathbb{C}$, and $P_f^{0,2}$ and $P_f^{1,1}$ are the projections to $(0,2)$ and $(1,1)$ -parts with respect to the almost complex structure f^*J . We denote the left hand side of (2.5) by $(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^*)(\alpha + \beta)$.

In Section 4, we prove that there are no rank one or two solutions $(A, (\alpha, \beta))$ to the equations (2.5) and (2.6) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $(\alpha, \beta) \neq 0$ for a generic choice of perturbation parameters.

As in [F, Lem. 3.2], the differential of the above perturbed Dirac operator $(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^*)$ is given by

$$\begin{aligned} D \left(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^* \right)_{(A,(f,\theta))} (a, \underline{f}, \underline{\theta})(\mathfrak{a} + \mathfrak{b}) \\ = \sum \underline{f}(v^i) \wedge \nabla_{A,v_i} \mathfrak{a} - \sum \iota(\underline{f}(v^i)) \nabla_{A,v_i} \mathfrak{b} \\ + \rho(f(a))(\mathfrak{a} + \mathfrak{b}) + \rho(f(\underline{\theta}))(\mathfrak{a} + \mathfrak{b}), \end{aligned}$$

where $a \in \Omega^1(\mathfrak{g}_P)$, $\underline{f} \in C^r(\mathfrak{gl}(T^*X))$, $\underline{\theta} \in C^r(\Lambda^1 \otimes \mathbb{C})$ and $\mathfrak{a} \in \Omega^0(\mathfrak{g}_P)$, $\mathfrak{b} \in \Omega^{0,2}(\mathfrak{g}_P)$.

3 Generic smoothness for the rank three case

In this section, we prove Proposition 1.3. In order to do that, we consider the *parametrized moduli space*, and prove that it is a smooth manifold (Proposition 3.1). Then Proposition 1.3 follows from Proposition 3.1.

3.1 Parametrized moduli space

Let X be a compact symplectic four-manifold with symplectic form ω , and let P be a principal G -bundle over X . From now on, G is either $SU(2)$ or $SO(3)$. We take an almost complex structure J compatible with the symplectic form ω .

We denote by $\mathcal{A}_k^2(P)$ the L_k^2 -completion of the space of connections on P , and by $\mathcal{G}(P) = \mathcal{G}_{k+1}^2(P)$ the L_{k+1}^2 -completion of the gauge group. We then put

$$\mathcal{C}(P) := \mathcal{A}_k^2(P) \times L_k^2(\mathfrak{g}_P \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2})).$$

We next put $\mathcal{P}_1 := C^r(GL(\Lambda^{0,2})) \times C^r(GL(\Lambda^{0,0})) \times C^r(GL(\Lambda^{2,2})) \times C^r(\Lambda^1 \otimes \mathbb{C})$. This is the parameter space for the perturbation described in Section 2.2. The action of $\mathcal{G}(P)$ on \mathcal{P}_1 is considered to be trivial. We denote the quotient $\mathcal{C}(P)/\mathcal{G}(P)$ by $\mathcal{B}(P)$.

We define

$$s : \mathcal{C}(P) \times \mathcal{P}_1 \rightarrow L_{k-1}^2(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda^{0,2} \oplus \Lambda^{1,1}))$$

by $s(A, (\alpha, \beta), \tau, \theta) := (s_1(A, (\alpha, \beta), \tau, \theta), s_2(A, (\alpha, \beta), \tau, \theta))$, where

$$\begin{aligned} s_1(A, (\alpha, \beta), \tau, \theta) &:= \bar{\partial}_A \alpha + \bar{\partial}_A^* \beta + \rho(\theta)(\alpha + \beta), \\ s_2(A, (\alpha, \beta), \tau, \theta) &:= F_A^{0,2} + \frac{1}{2} \tau_1 [\alpha, \beta] + \Lambda F_A^{1,1} \wedge \omega \\ &\quad + \frac{1}{2} \tau_2 [\alpha, \alpha^*] \wedge \omega + \Lambda \tau_3 [\beta, \beta^*]. \end{aligned}$$

This is a $\mathcal{G}(P)$ -equivariant map. Here $\rho : T^*X \otimes \mathbb{C} \rightarrow \text{Hom}_{\mathbb{C}}(\Lambda^{0,0} \otimes \Lambda^{0,2}, \Lambda^{0,1})$ is the Clifford multiplication, and $\tau := (\tau_1, \tau_2, \tau_3) \in \mathcal{P}_1$. We call $M(P) := s^{-1}(0)/\mathcal{G}(P) \subset \mathcal{B}(P) \times \mathcal{P}_1$ the *parametrized moduli space*.

We denote by $\mathcal{B}_{\diamond}^*(P)$ gauge equivalence classes of pairs $(A, (\alpha, \beta)) \in \mathcal{C}(P)$ with A irreducible, $\alpha - \bar{\alpha} = 0$ and $\alpha + \beta$ being of rank three. We put $M_{\diamond}^*(P) := M(P) \cap (\mathcal{B}_{\diamond}^*(P) \times \mathcal{P}_1)$. We then have the following.

Proposition 3.1. *The zero set $s^{-1}(0)$ in $\mathcal{B}_{\diamond}^*(P) \times \mathcal{P}_1$ is regular, in particular, the parametrized moduli space $M_{\diamond}^*(P)$ is a smooth Banach submanifold of $\mathcal{B}_{\diamond}^*(P) \times \mathcal{P}_1$.*

We prove Proposition 3.1 in Section 3.2. Proposition 1.3 follows from Proposition 3.1 as described below.

Proof of Proposition 1.3. Note that s is a Fredholm section if it is restricted to $\mathcal{B}(P) \times \{(\tau, \theta)\}$ for a perturbation parameter (τ, θ) . Thus, by the Sard–Smale theorem ([DK, Prop 4.3.11]), there exists a first category subset \mathcal{P}'_1 such that the zero set of s in $\mathcal{B}_{\diamond}^*(P)$ is regular for $(\tau, \theta) \in \mathcal{P}_1 \setminus \mathcal{P}'_1$. Hence, $\mathcal{M}_{\diamond}^*(\tau, \theta) = s^{-1}(0) \cap \mathcal{B}_{\diamond}^*(P)$ is a smooth manifold for generic C^r -parameters (τ, θ) . \square

3.2 Proof of Proposition 3.1

In this section, we prove Proposition 3.1. We follow an argument by Feehan [F, §2.2] (see also [DK, §4.3.5]). Firstly, we consider the linearization $Ds = (Ds_1, Ds_2) : L_k^2(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times L_k^2(\mathfrak{g}_P \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2})) \times \mathcal{P}_1 \rightarrow L_{k-1}^2(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda^{1,1} \oplus \Lambda^{0,2}))$ of s at $(A, (\alpha, \beta), \tau, \theta) \in s^{-1}(0)$, where

$$\begin{aligned} Ds_1((\underline{\tau}, \underline{\theta}), a, (\mathfrak{a}, \mathfrak{b})) &= \bar{\partial}_A \mathfrak{a} + \bar{\partial}_A^* \mathfrak{b} + \rho(\theta)(\mathfrak{a} + \mathfrak{b}) + \rho(\underline{\theta})(\alpha + \beta), \\ Ds_2((\underline{\tau}, \underline{\theta}), a, (\mathfrak{a}, \mathfrak{b})) &= \bar{\partial}_A a + \partial_A a + \frac{1}{2} \tau_1 ([\mathfrak{a}, \beta] + [\alpha, \mathfrak{b}]) + \frac{1}{2} \tau_1 \tau_1 [\alpha, \beta] \\ &\quad + \frac{1}{2} \tau_2 ([\alpha, \mathfrak{a}^*] + [\mathfrak{a}, \alpha^*]) \wedge \omega + \frac{1}{2} \tau_2 \tau_2 ([\alpha, \alpha^*]) \wedge \omega \\ &\quad + \Lambda \tau_3 ([\beta, \mathfrak{b}^*] + [\mathfrak{b}, \beta^*]) + \Lambda \tau_3 \tau_3 ([\beta, \beta^*]). \end{aligned}$$

We then suppose for a contradiction that there exists $(\delta, v) \in C^0(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times C^0(\mathfrak{g}_P \otimes (\Lambda^{1,1} \oplus \Lambda^{0,2}))$ with $(\delta, v) \neq 0$ such that

$$\langle Ds_1(a, (\mathfrak{a}, \mathfrak{b}), \underline{\tau}, \underline{\theta}), \delta \rangle_{L^2} = 0, \quad \langle Ds_2(a, (\mathfrak{a}, \mathfrak{b}), \underline{\tau}, \underline{\theta}), v \rangle_{L^2} = 0. \quad (3.1)$$

By putting $(\mathfrak{a}, \mathfrak{b}) = 0$ in the first equation of (3.1), we get

$$\langle \rho(\underline{\theta})(\alpha + \beta), \delta \rangle_{L^2} = 0 \quad (3.2)$$

for $\underline{\theta} \in C^r(\Lambda^1 \otimes \mathbb{C})$.

Lemma 3.2. *Assume that $\alpha + \beta \in C^0(\mathfrak{g}_P \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2}))$ and $\delta \in C^0(\mathfrak{g}_P \otimes \Lambda^{0,1})$ satisfy (3.2). Then $\alpha + \beta$ and δ have orthogonal images in \mathfrak{g}_P at each point of X , in particular,*

$$\text{rank}_{\mathbb{R}}(\alpha + \beta)(x) + \text{rank}_{\mathbb{R}} \delta(x) \leq 3$$

at each point $x \in X$.

proof. In (3.2), $\underline{\theta} \in C^r(\Lambda^1 \otimes \mathbb{C})$ is arbitrary, thus, we get the pointwise identity

$$\langle \rho(\underline{\theta}_x)(\alpha + \beta)(x), \delta(x) \rangle_x = 0$$

for all $\underline{\theta}_x \in (T^*X)_x \otimes \mathbb{C}$.

We then recall the following.

Lemma 3.3 ([F], Lem. 2.3). *Let U and V be complex vector spaces with $\dim U \leq \dim V$, and let W be a real vector space. We take $M \in U^* \otimes_{\mathbb{R}} W$ and $N \in V^* \otimes_{\mathbb{R}} W$. Then, if $\langle MP, N \rangle_{V^* \otimes_{\mathbb{R}} W} = 0$ for all $P \in \text{Hom}_{\mathbb{C}}(V, U)$, we get $\text{Ran } M \perp \text{Ran } N$ in W , in particular, $\text{rank}_{\mathbb{R}} M + \text{rank}_{\mathbb{R}} N \leq \dim_{\mathbb{R}} W$.*

Since ρ gives a complex linear isomorphism

$$(T^*X)_x \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \text{Hom}_{\mathbb{C}}(\Lambda^{0,1} \oplus \Lambda^{0,2}, \Lambda^{0,1})_x,$$

we can invoke Lemma 3.3 to obtain the assertion. \square

As $(A, (\alpha, \beta)) \in \mathcal{C}_{\diamond}^*(P)$ and $\alpha + \beta$ is C^r for some r , there is a non-empty open subset $U \subset X$ on which $\text{rank}_{\mathbb{R}}(\alpha + \beta)(x) = 3$ for all $x \in U$. Then Lemma 3.2 implies that $\text{rank } \delta(x) = 0$ for all $x \in U$, namely, $\delta \equiv 0$ on U .

In a similar way, by putting $(a, (\mathfrak{a}, \mathfrak{b})) = 0$ in the second equation of (3.1), we get

$$\left\langle \frac{1}{2}\underline{\tau}_1\tau_1[\alpha, \beta] + \frac{1}{2}\underline{\tau}_2\tau_2[\alpha, \alpha^*] \wedge \omega + \Lambda\underline{\tau}_3\tau_3[\beta, \beta^*], v \right\rangle_{L^2(X)} = 0 \quad (3.3)$$

for all $\underline{\tau}_1 \in C^r(\mathfrak{gl}(\Lambda^{0,2}))$, $\underline{\tau}_2 \in C^r(\mathfrak{gl}(\Lambda^{0,0}))$ and $\underline{\tau}_3 \in C^r(\mathfrak{gl}(\Lambda^{2,2}))$.

Lemma 3.4. *If $v \in C^0(\mathfrak{g}_P \otimes (\Lambda^{1,1} \oplus \Lambda^{0,2}))$ and $\alpha + \beta \in C^0(\mathfrak{g}_P \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2}))$ satisfy (3.3), then v and $\frac{1}{2}\tau_1[\alpha, \beta] + \frac{1}{2}\tau_2[\alpha, \alpha^*] \wedge \omega + \Lambda\tau_3[\beta, \beta^*] \in \text{Hom}((\Lambda^{1,1} \oplus \Lambda^{0,2})^*, \mathfrak{g}_P)$ have orthogonal images in \mathfrak{g}_P at each point in X , in particular,*

$$\text{rank}_{\mathbb{R}} v(x) + \text{rank}_{\mathbb{R}} \left(\frac{1}{2}[\alpha, \beta] + \frac{1}{2}[\alpha, \alpha^*] \wedge \omega + \Lambda[\beta, \beta^*] \right)(x) \leq 3$$

at each $x \in X$.

proof. As $\underline{\tau}_1 \in C^r(\mathfrak{gl}(\Lambda^{0,2}))$, $\underline{\tau}_2 \in C^r(\mathfrak{gl}(\Lambda^{0,0}))$ and $\underline{\tau}_3 \in C^r(\mathfrak{gl}(\Lambda^{2,2}))$ are arbitrary, we get the following pointwise identity.

$$\left\langle \left(\frac{1}{2}\underline{\tau}_1\tau_1[\alpha, \beta] + \frac{1}{2}\underline{\tau}_2\tau_2[\alpha, \alpha^*] \wedge \omega + \Lambda\underline{\tau}_3\tau_3[\beta, \beta^*] \right)(x), v(x) \right\rangle_x = 0$$

for all $\underline{\tau}_1(x) \in C^r(\mathfrak{gl}(\Lambda^{0,2}|_x))$, $\underline{\tau}_2(x) \in C^r(\mathfrak{gl}(\Lambda^{0,0}|_x))$, $\underline{\tau}_3(x) \in C^r(\mathfrak{gl}(\Lambda^{2,2}|_x))$ and for all $x \in X$. Then we again invoke Lemma 3.3 to obtain the assertion. \square

The following is due to Mares [Ma, §4.1.1].

Lemma 3.5 ([Ma]). *Let $(A, \alpha + \beta)$ be an irreducible solution to the equation, and let $x \in X$. Then $\text{rank}_{\mathbb{R}} \left(\frac{1}{2}[\alpha, \beta] + \frac{1}{2}[\alpha, \alpha^*] \wedge \omega + \Lambda[\beta, \beta^*] \right)(x) = 3$ if and only if $\text{rank}_{\mathbb{R}}(\alpha + \beta)(x) = 3$.*

From Lemma 3.5, if $\text{rank}(\alpha + \beta)(x) = 3$ for all $x \in U$, then $\text{rank}_{\mathbb{R}} \frac{1}{2}[\alpha, \beta] + \frac{1}{2}[\alpha, \alpha^*] \wedge \omega + \Lambda[\beta, \beta^*](x) = 3$ for all $x \in U$. Thus Lemma 3.4 implies $\text{rank } v(x) = 0$ for all $x \in U$. Therefore, $v \equiv 0$ on U . Hence $(\delta, v) \equiv 0$ on U . Thus by unique continuation for the Laplacian $(Ds)(Ds)^*$ implies that $(\delta, v) \equiv 0$ on the whole of X . This is a contradiction. \square

4 Non-existence of rank one and two cases

In this section, we prove Proposition 1.4. Except modifications stated as Proposition 4.2 in Section 4.2 and Proposition 4.4 in Section 4.3, the proof goes in a similar way to that for the $PU(2)$ -monopole equation by Feehan [F]. In Section 4.1, we introduce some terminology and a version of the Sard–Smale theorem from [F] needed in the later sections. The characterization of the rank one and two sections is given in Section 4.2. In Section 4.3, a surjectivity of some linear operator is proved. In Section 4.4, we prove Proposition 1.4 by using the Sard–Smale theorem.

4.1 Banach spaces, Fredholm operators and the Sard–Smale theorem

Let V be a Banach space. For each $k \geq 1$, we define the following infinite dimensional Grassmannian by

$$\mathbb{G}_k(V) := \{K \subset V : K \text{ is a } k\text{-dimensional subspace of } V\}.$$

We write $\mathbb{P}(V) = \mathbb{G}_1(V)$. We also define an infinite dimensional flag manifold by

$$\mathbb{F}_k(V) := \{(\ell, K) \in \mathbb{P}(V) \times \mathbb{G}_k(V) : \ell \subset K\}.$$

We denote the projections by $\pi_1 : \mathbb{F}_k(V) \rightarrow \mathbb{P}(V)$ and $\pi_2 : \mathbb{F}_k(V) \rightarrow \mathbb{G}_k(V)$. Both π_1 and π_2 are submersions (see Claims 4.2 and 4.3 in [F]).

We next consider a smooth submanifold $Z \in \mathbb{P}(V)$. We define $I_k(Z) := \pi_2(\pi_1^{-1}(Z)) \subset \mathbb{G}_k(V)$. As π_1 is a submersion, $\tilde{I}_k(Z) := \pi_1^{-1}(Z) \subset \mathbb{F}_k(V)$ is a smooth submanifold. Note that, however, $I_k(Z)$ is not necessarily a submanifold.

Space of Fredholm operators. Let V_1, V_2 be Banach spaces. We denote by $\text{Fred}_n(V_1, V_2)$ the space of bounded Fredholm operators of index n in the Banach space of the bounded operators. In our case, we take $V_1 := L_k^2(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2}))$, where $\Lambda_I^{0,0} := \{\alpha \in \Lambda^{0,0} : \alpha - \bar{\alpha} = 0\}$ and $V_2 := L_{k-1}^2(\mathfrak{g}_P \otimes \Lambda^{0,1})$ in the subsequent sections. We put

$$\text{Fred}_{k,n} := \{A \in \text{Fred}_n(V_1, V_2) : \dim_{\mathbb{R}} \ker A = k\}.$$

We define a map

$$\pi : \text{Fred}_{k,n}(V_1, V_2) \rightarrow \mathbb{G}_k(V_1)$$

by $A \mapsto \ker A$. This is a smooth map, and a submersion ([F, Lem. 4.5]). We also define the following flag manifold for each $\text{Fred}_{k,n}(V_1, V_2)$.

$$\text{Flag}_{k,n}(V_1, V_2) = \{(\ell, A) \in \mathbb{P}(V_1) \times \text{Fred}_{k,n}(V_1, V_2) : \ell \in \ker A\}.$$

This $\text{Flag}_{k,n}(V_1, V_2)$ is a smooth submanifold of $\mathbb{P}(V_1) \times \text{Fred}_{k,n}(V_1, V_2)$ and the canonical map $\varpi : \text{Flag}_{k,n}(V_1, V_2) \rightarrow \mathbb{P}(V_1)$ is a submersion (see [F, Lem. 4.6]).

The Sard–Smale theorem. We state a version of the Sard–Smale theorem from [F].

Proposition 4.1 ([F], Prop.4.12). *Let $\mathcal{C}, \mathcal{P}, \mathcal{F}$ be C^∞ -Banach manifolds. Suppose that $M \subset \mathcal{C} \times \mathcal{P}$ is a C^∞ -Banach submanifold, and the restriction $\pi_{M,\mathcal{P}} : M \rightarrow \mathcal{P}$ of the projection map $\pi_{\mathcal{P}} : \mathcal{C} \times \mathcal{P} \rightarrow \mathcal{P}$ is Fredholm. Let $\underline{v} : M \subset \mathcal{C} \times \mathcal{P} \rightarrow \mathcal{F}$ be a C^∞ -map which is transverse to a C^∞ -Banach submanifold $\mathcal{J} \subset \mathcal{F}$. Then there exists a first category subset $\mathcal{P}' \subset \mathcal{P}$ such that the following holds. For all $p \in \mathcal{P} \setminus \mathcal{P}'$,*

- $M := \pi_{M,\mathcal{P}}^{-1}(p)$ is a C^∞ -manifold of dimension $\text{ind}(\pi_{M,\mathcal{P}})_p < \infty$;
- $v := \underline{v}(\cdot, p) : M \rightarrow \mathcal{F}$ is transverse to the submanifold $\mathcal{J} \subset \mathcal{F}$; and
- $Z := v^{-1}(\mathcal{J}) \subset M$ is a C^∞ -submanifold of codimension $\text{codim}(Z, M) = \text{codim}(\mathcal{J}, \mathcal{F})$.

We use this to prove Proposition 1.4 in Section 4.4.

4.2 Rank one and two loci

We take $k \geq 4$ so that $V_1 = L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2})) \subset C^0(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2}))$. We think of $C^0(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2}))$ as $C^0\left(\text{Hom}_{\mathbb{R}}\left((\Lambda_I^{0,0} \oplus \Lambda^{0,2})^*, \mathfrak{g}_P\right)\right)$, and define a determinant map

$$h : C^0(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2})) \rightarrow C^0\left(\det(\Lambda_I^{0,0} \oplus \Lambda^{0,2}) \otimes \det(\mathfrak{g}_P)\right)$$

by $\varphi \in C^0(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2})) \mapsto \det \varphi$, where $\det(\Lambda_I^{0,0} \oplus \Lambda^{0,2}) = \Lambda^3(\Lambda_I^{0,0} \oplus \Lambda^{0,2})$ and $\det(\mathfrak{g}_P) = \Lambda^3 \mathfrak{g}_P$. Then $\varphi \in V_1$ with $\varphi \neq 0$ is of rank one or two if and only if $h(\varphi) = 0$. We define

$$\mathcal{Z} := \{[\varphi] \in \mathbb{P}(V_1) : h(\varphi) = 0\},$$

where $[\varphi]$ is the line $\mathbb{R} \cdot \varphi \subset V_1$. We denote by \mathcal{Z}' the smooth part of \mathcal{Z} .

As in the case of the $PU(2)$ -monopole equation [F, Lem 4.7], one obtains the following.

Proposition 4.2. *Let $[\varphi] \in \mathcal{Z}$. We assume that $\{\varphi \neq 0\}$ is a dense open subset of X . Then the determinant map $h : C^0(\mathfrak{g}_P \otimes (\Lambda_I^{0,0} \oplus \Lambda^{0,2})) \rightarrow C^0(\det(\Lambda_I^{0,0} \oplus \Lambda^{0,2}) \otimes \det(\mathfrak{g}_P))$ vanishes transversely at φ , and $[\varphi]$ is a smooth point of \mathcal{Z} . In addition, the tangent space $T_{[\varphi]}\mathcal{Z}$ has both infinite dimension and infinite codimension in $T_{[\varphi]}\mathbb{P}(V_1)$, in particular, we have $\text{codim}(\mathcal{Z}', \mathbb{P}(V_1)) = \infty$.*

proof. We take a local orthonormal frame $\{\phi_1, \phi_2, \phi_3\}$ for \mathfrak{g}_P , and local orthonormal frame $\{e_1, e_2, e_3\}$ for $\Lambda_I^{0,0} \oplus \Lambda^{0,2}$ on an open subset $U \subset X$ so

that $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix}$. Then the differential of h at φ with respect to these frame is given by

$$(Dh)_\varphi(\underline{\varphi}) = \sum_{\sigma \in \mathfrak{S}_3} \left\{ \text{sgn}(\sigma) \left(\varphi_{1\sigma(1)} \varphi_{2\sigma(2)} \varphi_{3\sigma(3)} + \varphi_{1\sigma(1)} \underline{\varphi_{2\sigma(2)}} \varphi_{3\sigma(3)} + \varphi_{1\sigma(1)} \varphi_{2\sigma(2)} \underline{\varphi_{3\sigma(3)}} \right) \right\},$$

where $\underline{\varphi} = \begin{pmatrix} \underline{\varphi_{11}} & \underline{\varphi_{12}} & \underline{\varphi_{13}} \\ \underline{\varphi_{21}} & \underline{\varphi_{22}} & \underline{\varphi_{23}} \\ \underline{\varphi_{31}} & \underline{\varphi_{32}} & \underline{\varphi_{33}} \end{pmatrix} \in C^\infty(U, \mathfrak{gl}(3, \mathbb{R}))$.

We now suppose for a contradiction that there exists $\psi \in \text{coker}(Dh)_\varphi$ so that $\langle (Dh)_\varphi(\underline{\varphi}), \psi \rangle_{L^2} = 0$ for all $\underline{\varphi} \in C^0(V_1)$. From the assumption, $\{\varphi \neq 0\}$ is dense in U , so the union of the complements of each zero set of $\varphi'_{ij,s}$ is a dense open subset of U , hence we get $\psi \equiv 0$ on U . Since U was arbitrary, $\psi \equiv 0$ on X . This is a contradiction. \square

We denote by $M^{*,0}(P)$ the parametrized moduli space for the perturbed Vafa–Witten equations (2.5) and (2.6) with A irreducible, $\alpha - \bar{\alpha} = 0$ and $(\alpha, \beta) \neq 0$. From Proposition 4.2, we get the following.

Corollary 4.3. *If $(A, \varphi = (\alpha, \beta), \tau, \theta)$ is in $M^{*,0}(P)$ so that $h(\varphi) = 0$, then $[\varphi]$ is a smooth point of $\mathcal{Z} \subset \mathbb{P}(V_1)$, that is, $\pi(M^{*,0}(P)) \subset \mathcal{Z}'$, where $\pi : M^{*,0}(P) \rightarrow \mathbb{P}(V_1)$ is the projection.*

For each $k \geq n$, we now define

$$\tilde{I}_k(\mathcal{Z}) := \pi_1^{-1}(\mathcal{Z}) \subset \mathbb{F}_k(V_1),$$

and $I_k(\mathcal{Z}) := \pi_2(\tilde{I}_k(\mathcal{Z})) \subset \mathbb{G}_k(V_1)$. By Corollary 4.3, we only consider $I_k(\mathcal{Z}')$ and $\tilde{I}_k(\mathcal{Z}')$ for our purpose. As $\pi_1 : \mathbb{F}_k(V) \rightarrow \mathbb{P}(V)$ is a submersion ([F, Claim 4.2]), $\tilde{I}_k(\mathcal{Z}')$ is a smooth submanifold of $\mathbb{F}_k(V_1)$ with codimension

$$\text{codim}(\tilde{I}_k(\mathcal{Z}'), \mathbb{F}_k(V_1)) = \text{codim}(\mathcal{Z}', \mathbb{P}(V_1)) = \infty.$$

We put $J_k(\mathcal{Z}') := \pi^{-1}(I_k(\mathcal{Z}')) \subset \text{Fred}_{k,n}(V_1, V_2)$, where $\pi : \text{Fred}_{k,n}(V_1, V_2) \rightarrow \mathbb{G}_k(V_1)$. We now define the *rank one and two loci* $\tilde{J}_k(\mathcal{Z}') := \varpi^{-1}(\tilde{I}_k(\mathcal{Z}')) \subset \text{Flag}_{k,n}(V_1, V_2)$, where $\varpi : \text{Flag}_{k,n}(V_1, V_2) \rightarrow \mathbb{F}_k(V_1)$ is the canonical map. As $\varpi : \text{Flag}_{k,n}(V_1, V_2) \rightarrow \mathbb{F}_k(V_1)$ is a submersion ([F, Lem 4.6]), the rank one and two loci $\tilde{J}_k(\mathcal{Z}')$ is a smooth submanifold, and we get

$$\text{codim}(\tilde{J}_k(\mathcal{Z}'), \text{Flag}_{k,n}(V_1, V_2)) = \text{codim}(\tilde{I}_k(\mathcal{Z}'), \mathbb{F}_k(V_1)) = \infty.$$

4.3 A surjectivity

In this section and the upcoming one, we take $\mathcal{P}_2 := C^r(GL(T^*X)) \times C^r(\Lambda^1 \otimes \mathbb{C})$ as the perturbation parameter space, since the perturbation parameter $\tau = (\tau_1, \tau_2, \tau_3)$ is not needed in the proof of Proposition 1.4.

We denote by $\mathcal{C}^*(P)$ the set of pairs $(A, (\alpha, \beta)) \in \mathcal{C}(P)$ with A irreducible and $\alpha - \bar{\alpha} = 0$. As in [F, §4.4] (see also [DK, §4.3.3]), we consider the following *period map*

$$v : \mathcal{C}^*(P) \times \mathcal{P}_2 \rightarrow \text{Fred}_n(V_1, V_2),$$

defined by $(A, (\alpha, \beta), f, \theta) \mapsto \left(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^* \right)$. The differential of v at $(A, (\alpha, \beta), f, \theta)$

$$(Dv)_{(A,(\alpha,\beta),f,\theta)} : T_{(A,(\alpha,\beta))}\mathcal{C}^*(P) \oplus T_{(f,\theta)}\mathcal{P}_2 \rightarrow \text{Hom}_{\mathbb{R}}(V_1, V_2)$$

is given by $(a, (\underline{\alpha}, \underline{\beta}), \underline{f}, \underline{\theta}) \mapsto D \left(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^* \right)_{(A,(f,\theta))} (a, \underline{f}, \underline{\theta})$.

We denote by $\mathcal{C}^{*,0}(P)$ the set of pairs $(A, (\alpha, \beta)) \in \mathcal{C}(P)$ with A irreducible, $\alpha - \bar{\alpha} = 0$ and $(\alpha, \beta) \neq 0$, and by $\mathcal{B}^{*,0}(P)$ the quotient $\mathcal{C}^{*,0}(P)/\mathcal{G}(P)$. We set $M^{*,0}(P) = M(P) \cap (\mathcal{B}^{*,0}(P) \times \mathcal{P}_2)$, where $M(P)$ is the parametrized moduli space for the equations (2.5) and (2.6). In this section, we prove the following.

Proposition 4.4. *Let $(A, (\alpha, \beta), f, \theta) \in M^{*,0}(P)$. Then, the following is surjective.*

$$(Dv)_{(A,(\alpha,\beta),f,\theta)}(0, \cdot) : \{0\} \oplus T_{(f,\theta)}\mathcal{P}_2 \rightarrow T_{v(A,(\alpha,\beta),f,\theta)}\text{Fred}_n(V_1, V_2).$$

proof. A proof here is a modification of that of [F, Prop. 4.9]. We first prove the following lemma.

Lemma 4.5. *Assume that $(A, (\alpha, \beta))$ is a solution to the Vafa–Witten equations (2.5) and (2.6) with A irreducible and $(\alpha, \beta) \neq 0$ for some perturbation parameter $(f, \theta) \in \mathcal{P}_2$. If $b \in \Omega^{0,0}(X, \mathfrak{g}_P) \oplus \Omega^{0,2}(X, \mathfrak{g}_P)$ and $d \in \Omega^{0,1}(X, \mathfrak{g}_P)$ satisfy*

$$\left\langle D \left(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^* \right)_{(A,(f,\theta))} (\underline{f}, \underline{\theta}), d \otimes b^* \right\rangle_{L^2(X)} = 0$$

for all $(\underline{f}, \underline{\theta})$, then $d \otimes b^* \equiv 0$ on X .

proof. Suppose for a contradiction that $d \otimes b^* \neq 0$ on X . By varying $\underline{\theta}$, we see that b and d have orthogonal images in \mathfrak{g}_P at each point $x \in X$ from Lemma 3.2. We then put $U := \{b \neq 0\} \cap \{d \neq 0\} \subset X$. Then either b or d defines a subbundle $\xi_1 \subset \mathfrak{g}_P$ on U of rank $_{\mathbb{R}} = 2$. We put $\xi_2 := \xi_1^\perp \subset \mathfrak{g}_P|_U$ so that $\mathfrak{g}_P|_U = \xi_1 \oplus \xi_2$. The connection $A|_U$ on $\mathfrak{g}_P|_U$ also splits into the following form.

$$A = \begin{pmatrix} A_1 & -\chi^* \\ \chi & A_2 \end{pmatrix},$$

where A_i is a connection on ξ_i for $i = 1, 2$, and $\chi \in \Omega^1(U, \xi_2 \otimes \xi_1^*)$ is the second fundamental form. As $(A, (\alpha, \beta))$ is irreducible and non-zero section from the assumption, $\chi \neq 0$ on $U \subset X$. We suppose that $b \in \Omega^{0,0}(U, \xi_1) \oplus \Omega^{0,2}(U, \xi_1)$. We then get

$$\begin{aligned} & D \left(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^* \right)_{(A,(f,\theta))} (\underline{f}, \underline{\theta}) b \\ &= \sum_{i=1}^4 (\underline{f}(v^i)) \wedge \nabla_{A_1, v_i} b - \sum_{i=1}^4 \iota(\underline{f}(v^i)) \nabla_{A_1, v_i} b + \rho(f(\underline{\theta})) b + \rho(\underline{f}(\chi)) b. \end{aligned}$$

This turns out to be

$$\left\langle \rho \left(\underline{f}_x(\chi_x) \right) b_x, d_x \right\rangle_x = 0$$

at each $x \in U$ and for all $\underline{f}_x \in \mathfrak{gl}(T^*X)_x$. Hence we get $d_x \otimes b_x^* = 0$ at each $x \in U$ with $\chi_x \neq 0$. As $d_x \otimes b_x^* \neq 0$ for all $x \in U$ from the assumption, we get $\chi = 0$, thus, $A|_U$ is reducible.

On the other hand, by a similar argument to that by Feehan–Lenes [FL, §5.3], one can obtain that, if A is reducible on a non-empty open subset $U \subset X$ and $(\alpha, \beta) \neq 0$, then A is reducible on X . This is a contradiction. Therefore, $U \subset X$ is empty and $d \otimes b^* \equiv 0$ on X . \square

We now suppose that $(Dv)_{(A,(\alpha,\beta),f,\theta)}(0, \cdot)$ is not surjective. Then, there exist sections $b \in L_k^2(V_1)$ and $d \in L_{k-1}^2(V_2)$ with $d \otimes b^* \not\equiv 0$ on X such that

$$\left\langle D \left(\bar{\partial}_{A,(f,\theta)} + \bar{\partial}_{A,(f,\theta)}^* \right)_{(A,(f,\theta))} (\underline{f}, \underline{\theta}) b, d \right\rangle = 0.$$

Then, from Lemma 4.5, we get $d \otimes b^* \equiv 0$. This is a contradiction. Therefore, $(Dv)_{(A,(\alpha,\beta),f,\theta)}(0, \cdot)$ is surjective. \square

4.4 No rank one and two loci

In this section, we prove Proposition 1.4. This will be done at the end of this section. As mentioned in the beginning of Section 4, once Propositions 4.2 and 4.4 are obtained, then the proof goes along the same line with that for the $PU(2)$ -monopole equation [F, §4.6]. Hence we give it sketchily.

Firstly, note that the map $s : \mathcal{C}^{*,0}(P) \times \mathcal{P}_2 \rightarrow L_{k-1}^2(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda^{0,2} \oplus \Lambda^{1,1}))$ is right semi-Fredholm, namely, the differential has closed range and finite dimensional cokernel. In particular,

$$\mathbb{H}_{(A,(\alpha,\beta),p)}^2 := (\text{Im}(Ds(\cdot, p)))_{(A,(\alpha,\beta))}^\perp$$

is a finite dimensional subspace of $L_{k-1}^2(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda^{0,2} \oplus \Lambda^{1,1}))$. We denote by $\Pi_{(A,(\alpha,\beta))}$ the L^2 -orthogonal projection from $L_{k-1}^2(\mathfrak{g}_P \otimes \Lambda^{0,1}) \times L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda^{0,2} \oplus \Lambda^{1,1}))$ to the $\text{Im}(Ds(\cdot, p))_{(A,(\alpha,\beta))}$.

Let $(c_0, p_0) \in M^{*,0}(P)$. We consider the following composition.

$$\Pi_{(c_0, p_0)} \circ s : \mathcal{B}^{*,0}(P) \times \mathcal{P}_2 \rightarrow \left(\mathbb{H}_{(c_0, p_0)}^2 \right)^\perp$$

Then the differential at (c_0, p_0) of $\Pi_{(c_0, p_0)} \circ s$ is surjective, in particular, it is surjective on some open neighbourhood $\mathcal{U}_{(c_0, p_0)}$ of (c_0, p_0) in $\mathcal{C}^{*,0}(P) \times \mathcal{P}_2$. We put

$$\mathcal{T}_{(c_0, p_0)} := \mathcal{U}_{(c_0, p_0)} \cap (\Pi_{(c_0, p_0)} \circ s)^{-1}(0) \subset \mathcal{B}^{*,0}(P) \times \mathcal{P}_2.$$

We denote by $\pi_{\mathcal{T}, \mathcal{P}_2} : \mathcal{T}_{(c_0, p_0)} \rightarrow \mathcal{P}_2$ the projection, and put $\mathcal{T}_{(c_0, p_0)}|_p := \pi_{\mathcal{T}, \mathcal{P}_2}^{-1}(p) \cap \mathcal{T}_{(c_0, p_0)}$. We then prove the following.

Proposition 4.6. *There is a first-category subset $\mathcal{P}_2' \subset \mathcal{P}_2$, depending on (c_0, p_0) such that for any $p \in \mathcal{P}_2 \setminus \mathcal{P}_2'$, $\mathcal{T}_{(c_0, p_0)}|_p$ contains no $(A, (\alpha, \beta), p)$ with $\alpha + \beta$ being of rank one nor two.*

proof. The argument consists of the following three steps. Firstly, we consider the period map v defined from $\mathcal{T}_{(c_0, p_0)}$ to $\text{Fred}_n(V_1, V_2)$. As the differential of v is not necessarily surjective, we *stabilize* the map to obtain a submersion $v' : \mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)} \rightarrow \text{Fred}_n(V_1, V_2)$, where $\mathcal{V}_{(c_0, p_0)}$ is some finite dimensional vector space in $T_{v(c_0, p_0)}\text{Fred}_n(V_1, V_2)$. Secondly, we lift the stabilized period map v' to $\mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)} \rightarrow \text{Flag}_{k, n}(V_1, V_2)$ as the rank one and two loci $\tilde{J}_k(Z')$ lives in $\text{Flag}_{k, n}(V_1, V_2)$. This is again not necessarily a submersion, so we stabilize it to obtain a smooth submersion $w' : \mathbb{C}^k \times \mathcal{W}_{(c_0, p_0), k} \rightarrow \text{Flag}_{k, n}(V_1, V_2)$, where $\mathcal{W}_{(c_0, p_0), k}$ is a submanifold of $\mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)}$ with finite codimension. Then we use the Sard–Smale theorem (Proposition 4.1) to the w' to obtain the assertion in Step 3.

Step 1. We first consider the period map $v : \mathcal{T}_{(c_0, p_0)} \rightarrow \text{Fred}_n(V_1, V_2)$. From Proposition 4.4, the operator

$$(Dv)_{(c_0, p_0)} : \{0\} \oplus T_{p_0}\mathcal{P}_2 \rightarrow T_{v(c_0, p_0)}\text{Fred}_n(V_1, V_2)$$

is surjective. On the other hand, we have

$$T_{(c_0, p_0)}\mathcal{T}_{(c_0, p_0)} + (\{0\} \oplus T_{p_0}\mathcal{P}_2) = \mathbb{H}_{(c_0, p_0)}^1 \oplus T_{p_0}\mathcal{P}_2,$$

where

$$\begin{aligned} \mathbb{H}_{(c_0, p_0)}^1 &:= \ker(Ds(\cdot, p_0))_{(c_0, p_0)} \\ &= \ker(\Pi_{(c_0, p_0)} \circ (Ds)(\cdot, p_0))_{(c_0, p_0)} \\ &= \ker(D\pi_{\mathcal{T}, \mathcal{P}_2})_{(c_0, p_0)} \subset T_{(c_0, p_0)}\mathcal{C}^{*, 0}(P). \end{aligned}$$

Hence, $(Dv)_{(c_0, p_0)} : \mathbb{H}_{(c_0, p_0)}^1 \oplus T_{p_0}\mathcal{P}_2 \rightarrow T_{v(c_0, p_0)}\text{Fred}_n(V_1, V_2)$ is surjective.

As [F, Lem 4.15], we also have the following isomorphism.

$$\left(\mathbb{H}_{(c_0, p_0)}^1 \oplus T_{p_0}\mathcal{P}_2\right) \cong T_{(c_0, p_0)}\mathcal{T}_{(c_0, p_0)} \oplus \text{coker}(D\pi_{\mathcal{T}, \mathcal{P}_2})_{(c_0, p_0)}.$$

We then define the following finite dimensional vector space.

$$V_{(c_0, p_0)} := (Dv)_{(c_0, p_0)}(\text{coker}(D\pi_{\mathcal{T}, \mathcal{P}_2})_{(c_0, p_0)}) \subset T_{v(c_0, p_0)}\text{Fred}_n(V_1, V_2).$$

We denote the inclusion by $i : V_{(c_0, p_0)} \rightarrow T_{v(c_0, p_0)}\text{Fred}_n(V_1, V_2)$. We then consider the following map.

$$v' : V_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)} \rightarrow \text{Fred}_n(V_1, V_2)$$

defined by $v'(y, (c, p)) := i(y) + v(c, p)$ for $(y, (c, p)) \in V_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)}$. As the differential of v' is surjective at $(0, c_0, p_0)$, there exists an open neighborhood of the origin $\mathcal{V}_{(c_0, p_0)} \subset V_{(c_0, p_0)}$ such that the restriction

$$v' : \mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)} \rightarrow \text{Fred}_n(V_1, V_2) \quad (4.1)$$

is a submersion.

We now consider the following for $k \geq n$.

$$\mathcal{W}_{(c_0, p_0), k} := (\mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)}) \cap (v')^{-1}(\text{Fred}_{k, n}(V_1, V_2)).$$

As (4.1) is a submersion, the above $\mathcal{W}_{(c_0, p_0), k}$ is a smooth submanifold with finite codimension in $\mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)}$, thus, $\mathcal{V}_{(c_0, p_0)} \times \mathcal{T}_{(c_0, p_0)} = \bigcup_{k \geq n} \mathcal{W}_{(c_0, p_0), k}$ is a countable disjoint union of smooth manifolds.

Step 2. We next lift the map $v' : \mathcal{W}_{(c_0, p_0), k} \rightarrow \text{Fred}_{k, n}(V_1, V_2)$ to a smooth map

$$w : \mathcal{W}_{(c_0, p_0), k} \rightarrow \text{Flag}_{k, n}(V_1, V_2)$$

by $(y, (A, (\alpha, \beta)), p) \mapsto ([(\alpha, \beta)], i(y) + v((A, (\alpha, \beta)), p))$. This is again not necessarily a submersion, so we stabilize it as described below.

Let $(y_1, (c_1, p_1))$ in $\mathcal{W}_{(c_0, p_0), k}$. Since a countable union of first category subsets is a first category subset; and $\mathcal{W}_{(c_0, p_0), k}$ is paracompact, we only consider a single open neighborhood of $(y_1, (c_1, p_1))$.

We take an orthonormal basis $\{b_{1, j}\}_{j=1}^k$ of the kernel of $v'(y_1, (c_1, p_1)) = i(y_1) + v(c_1, p_1)$. We denote by

$$\pi_{(y, (c, p))} : L_{k-1}^2(\mathfrak{g}_P \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2})) \rightarrow \ker(i(y) + v(c, p))$$

the smooth family of L^2 -orthogonal projection. We then consider a smooth map

$$w' : \mathbb{C}^k \times \mathcal{W}_{(c_0, p_0), k} \rightarrow \text{Flag}_{k, n}(V_1, V_2)$$

defined by

$$w'(z, y, c, p) \mapsto ([(\alpha, \beta) + \pi_{(y, (c, p))}(\sum_{j=1}^k z_j b_{1, j})], i(y) + v(c, p)),$$

where $z = (z_1, \dots, z_k) \in \mathbb{C}^k$. As [F, Claim 4.18], the map w' is a submersion at $(0, y_1, (c_1, p_1))$, thus $\mathcal{W}'_{(c_0, p_0), k} := (w')^{-1}(\tilde{J}_k(Z'))$ is a C^∞ -Banach submanifold of $\mathbb{C}^k \times \mathcal{W}_{(c_0, p_0), k}$.

Step 3. We are now in a situation to invoke the Sard–Smale theorem (Prop. 4.1). Applying it to w' , we obtain a first category subset $\mathcal{P}'_2 \subset \mathcal{P}_2$ such that for $p \in \mathcal{P}_2 \setminus \mathcal{P}'_2$

$$\text{codim}_{\mathbb{R}} \left(\mathcal{W}'_{(c_0, p_0), k}|_p, \mathbb{C}^k \times \mathcal{W}_{(c_0, p_0), k}|_p \right) = \text{codim}_{\mathbb{R}} \left(\tilde{J}_k(\mathcal{Z}'), \text{Flag}_{k, n}(V_1, V_2) \right).$$

Since $\text{codim}_{\mathbb{R}} \left(\tilde{J}_k(\mathcal{Z}'), \text{Flag}_{k, n}(V_1, V_2) \right) = \infty$ but $\dim_{\mathbb{R}} (\mathbb{C}^k \times \mathcal{W}_{(c_0, p_0), k}|_p) < \infty$, we deduce that $\mathcal{W}'_{(c_0, p_0), k}|_p$ is empty.

We also have $\mathcal{T}_{(c_0, p_0)}|_p \cap w|\mathcal{T}_{(c_0, p_0)}(\cdot, p)^{-1}(\tilde{J}_k(\mathcal{Z}')) \subset \mathcal{W}'_{(c_0, p_0), k}|_p$. Since $\mathcal{W}'_{(c_0, p_0), k}|_p$ is empty, thus so is $\mathcal{T}_{(c_0, p_0)}|_p \cap w|\mathcal{T}_{(c_0, p_0)}(\cdot, p)^{-1}(\tilde{J}_k(\mathcal{Z}'))$. Hence $\mathcal{T}_{(c_0, p_0)}$ has no rank one or two section $\alpha + \beta$ for $\dim \ker(\bar{\partial}_{A, p} + \bar{\partial}_{A, p}^*) = k$ and $p \in \mathcal{P}_2 \setminus \mathcal{P}'_2$. Since a countable union of first category subsets is a first category subset, by repeating this for $k \geq n$, we get the assertion. \square

Proof of Proposition 1.4. By Proposition 4.6, $M^{*,0}(P) \cap \mathcal{T}_{(c_0, p_0)} \subset \mathcal{T}_{(c_0, p_0)}$ has no rank one nor two solution $(A, (\alpha, \beta), p)$ for $p \in \mathcal{P}_2 \setminus \mathcal{P}'_2$. we then do this argument for each $(A, (\alpha, \beta), p) \in \mathcal{C}^{*,0}(P) \times \mathcal{P}_2$ to get an open neighborhood of it with no rank one or two sections for a first category subset. As $\mathcal{C}^{*,0}(P) \times \mathcal{P}_2$ is paracompact, we can cover $M^{*,0}(P)$ by countable such open neighborhoods. Since a countable union of first category subsets of \mathcal{P}_2 is again a first category subset of \mathcal{P}_2 , we get the assertion. \square

References

- [DK] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, 1990.
- [F] P. M. N. Feehan, *Generic metrics, irreducible rank-one $PU(2)$ monopoles, and transversality*, Comm. Anal. Geom. **8** (2000), 905–967.
- [FL] P. M. N. Feehan and T. G. Leness, *$PU(2)$ monopoles. I : Regularity, Uhlenbeck compactness, and transversality*, J. Differential Geom. **49** (1998), 265–410.
- [Ha] A. Haydys, *Fukaya–Seidel category and gauge theory*, arXiv:1010.2353.
- [LM] H. B. Lawson and M-L. Michaelshon, *Spin geometry*, Princeton University Press, Princeton, 1989.

- [Ma] B. Mares, *Some Analytic Aspects of Vafa–Witten Twisted $\mathcal{N} = 4$ Supersymmetric Yang–Mills theory*, Ph.D thesis, M.I.T., 2010.
- [Mo] J. W. Morgan, *The Seiberg–Witten equations and application to the topology of smooth four-manifolds*, Princeton Univ. Press, Princeton, 1994.
- [T1] Y. Tanaka, *Some boundedness property of the Vafa–Witten equations on closed four-manifolds*, arXiv1308.0862v3, preprint, 2014.
- [VW] C. Vafa and E. Witten, *A strong coupling test of S -duality*, Nucl. Phys. B, **432**, (1994), 484–550.
- [W] E. Witten, *Fivebranes and Knots*, Quantum Topol. **3** (2012), 1–137.

National Center for Theoretical Sciences (South), National Cheng Kung University, Tainan 701, Taiwan
yu2tanaka@gmail.com